ON THE DIMENSION OF THE DISCRETE WIGNER-VILLE TRANSFORM RANGE SPACE

Application to time-frequency based detectors design

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ABSTRACT

The information conveyed by the discrete Wigner-Ville representations of real, complex or analytic signals is highly redundant, each time-frequency location being related to others via non-obvious relationships. In this paper, we demonstrate that there also exists a large amount of linear relationships between time-frequency samples. This implies that a whole discrete Wigner-Ville representation can be determined from linear combinations of some selected time-frequency locations. A simple example illustrates this property. Next, we design a linear detector that only exploits the information provided by these locations and that yields the same performance as linear receivers performing in the whole time-frequency domain. Finally, some potential implications of this property are briefly presented.

1. INTRODUCTION

The discrete Wigner-Ville (DWV) distribution is of great interest for non-stationary signal analysis, synthesis and processing due to the numerous properties it satisfies [1-3]. However, it encodes the information in a redundant fashion since it maps a N-sample signal into a N by N representation. This increased amount of data often prohibits the use of the DWV distribution in applications like signal detection and classification.

In this paper, we demonstrate that it exists a large amount of linear relationships between TF samples. This means that the whole DWV representation of a real, complex or analytic signal can be evaluated from linear combinations of some selected locations. The use of this property may result in efficient algorithms as an example, since it appears that the number of these remarkable TF locations is about the quarter of the representation size for analytic signals. To illustrate this property, the design of a linear detector that only exploit the information available in such TF area is presented. Its performance is the same as classical linear receivers performing in the whole TF domain.

This paper is organized as follows. First, we briefly describe the problem. Next, we discuss the issue of obtaining linearly-independent locations in the DWV representations when signals are real, complex, or analytic. Then, this property is illustrated with the design of linear detectors performing in such area of the TF domain. Finally, we present some conclusions regarding other potential implications of this property.

2. DESCRIPTION OF THE PROBLEM

Let $\mathbf{x}=[x_0,\dots,x_{N-1}]^{\mathrm{T}}$ be a N-sample signal taken from a given linear signal space E. The DWV of $\mathbf{x}$ is a real valued distribution defined as follows:

$$\mathrm{W}_{\mathrm{x}}(n,m)=2\sum_{k=1-N/2}^{N/2-1} x_{n+k}\,x_{n-k}^{*}\exp\left(-\frac{4j\pi mk}{N}\right), \tag{1}$$

where $(m,n)\in[0,\dots,N\text{-}1]^2$.

From the above definition, we get the following sequence of maps:

$$\mathbf{x}\to\mathbf{R}_{\mathrm{x}}=\left[\dots x_{n+k}\,x_{n-k}^{*}\dots\right]^{\mathrm{T}},\ (n+k,n-k)\in[0,\dots,N-1]^2 \tag{2}$$

$$\mathbf{R}_{\mathrm{x}}\to\mathbf{W}_{\mathrm{x}}=\left[\dots\mathrm{W}_{\mathrm{x}}(n,m)\dots\right]^{\mathrm{T}},\ (n,m)\in[0,\dots,N-1]^2 \tag{3}$$

The number N_{R} of components of $\mathbf{R}_{\mathrm{x}}$ is as follows, if the length N of the signal $\mathbf{x}$ is even:

$$N_{\mathrm{R}}=\mathrm{card}\left\{(n+k,n-k)\in[0,\dots,N-1]^2\right\}=N^2/2,$$

Let F and G be the following sets:

$$F=\left\{\mathbf{R}_x\in\mathscr{C}^{N^2/2}:\mathbf{x}\in E\right\},$$

$$G=\left\{\mathbf{W}_x\in\mathscr{R}^{N^2}:\mathbf{x}\in E\right\}.$$

One of the objectives of this paper is to determine the number of linear relationships between the components of the vectors $\mathbf{W}_{\mathrm{x}}$ of G. Here, this set is referred to as *the range space of the DWV transform.*

It is noteworthy that F and G are not linear spaces (e.g. the sum of two DWV transforms is not a valid DWV transform). However, we define the dimension of the set F (resp. G), denoted dim(F) (resp. dim(G)), as the number of linearly-independent vectors that can generate the elements of F (resp. G) via linear combinations. Note that dim(G) is equal to the number of linearly-independent TF locations available in the DWV representations.

Since Eq. (1) matches the form of a discrete Fourier transform, F and G are isomorphic according to Eq. (3). This implies that the dimension of G is equal to the dimension of F. In Section 3, dim(F) is evaluated in the case of real complex and analytic signals.

3. DIMENSION OF THE DWV RANGE SPACE

3.1. Real signals (sketch)

Let $\mathbf{x}=[x_0,\ldots,x_{N-1}]^{\mathrm{T}}$ be a N-sample real signal. N is supposed to be even. It can be shown that the only linear relationships that exist between the components of $\mathbf{R}_x$ are those between $x_{n+k}x_{n-k}$, $k<0$, and $x_{n+k}x_{n-k}$, $k>0$. This implies that

$$\begin{aligned}\dim(F) &= \mathrm{card}\left\{(n+k,n-k)\in[0,\ldots,N-1]^2 : k\le 0\right\}\\ &= N^2/4+N/2.\end{aligned} \quad (4)$$

Consequently, the whole TF representation of a real signal can be generated from $N^2/4+N/2$ linearly-independent locations.

3.2. Non analytic complex signals (sketch)

Let $\mathbf{x}$ be a N-sample complex signal. N is supposed to be even. One can show that all the components of $\mathbf{R}_x$ can be generated from the real and imaginary parts of the cross products $x_{n+k}x_{n-k}^*$, $k\le 0$, that are linearly-independent. This implies that F is generated by the following number of linearly-independent vectors:

$$\begin{aligned}\dim(F) &= 2\,\mathrm{card}\left\{(n+k,n-k)\in[0,\ldots,N-1]^2 : k<0\right\}+N\\ &= N^2/2\end{aligned} \quad (5)$$

Note that the N cross-products $x_n x_n^*$ are members of $\mathcal{R}$ and that $x_{n+k}x_{n-k}^*$, $k<0$, are members of $\mathcal{C}$. This justifies the factor of 2 in Eq. (5) since $\mathcal{C}$ is isomorphic to $\mathcal{R}^2$. Consequently, the DWV representation of a non-analytic complex signal can be generated from $N^2/2$ linearly-independent TF locations.

3.3. Analytic signals (sketch)

The case of analytic signals is the most interesting since the Hilbert transform makes the negative frequencies content of real signals vanish, which yields a severe decrease of the number of interference terms [2].

Let $\mathbf{x}=[x_0,\ldots,x_{N-1}]^{\mathrm{T}}$ be a N-sample analytic signal. N is supposed to be even. Let $\mathbf{y}=[y_0,\ldots,y_{N/2},0,\ldots,0]^{\mathrm{T}}$ be the Fourier transform of $\mathbf{x}$ and H the following set:

$$H=\left\{\mathbf{R}_y=[\ldots,y_n y_m^*,\ldots]^{\mathrm{T}} : (n,m)\in[0,\ldots,N/2]^2\right\},$$

It can be shown that the components of $\mathbf{R}_y$ can be generated from the real and imaginary parts of the cross products $y_n y_m^*$, $m\le$ n, that are linearly-independent. This implies that the dimension of the set H is as follows:

$$\begin{aligned}\dim(H) &= 2\,\mathrm{card}\left\{(m,n)\in[0,\ldots,N/2]^2-(0,N/2):\right.\\ &\qquad \left.0\le m\le N/2,\, n>m\right\}+(N/2+1)+1\\ &= N^2/4+N.\end{aligned} \quad (6)$$

In Eq. (6), the additive coefficients $N/2+1$ and 1 respectively correspond to the cross-products $y_m y_m^*$, $0\le m\le N/2$, and $y_0 y_{N/2}^*$ that are real. Note that y_0 and $y_{N/2}$ are real because $\mathbf{x}$ is analytic.

Let $\underline{\mathbf{R}}_y$ the vector made up of the non-zero real and imaginary parts of the linearly-independent cross-products $y_n y_m^*$, $m\le$ n. From Eq. (6), $\underline{\mathbf{R}}_y$ is a member of $\mathcal{R}^{N^2/4+N}$. The vector $\mathbf{R}_x$ can be evaluated from $\underline{\mathbf{R}}_y$ as follows: $\mathbf{R}_x=\mathbf{A}\,\underline{\mathbf{R}}_y$, where $\mathbf{A}$ is $N^2/2$ by $N^2/4+N$ complex matrix, since we have:

$$x_{n+k}x_{n-k}^* = \frac{1}{N^2}\sum_{p=0}^{N/2}\sum_{q=0}^{N/2} y_p\, y_q^* \exp\left(2j\pi\frac{(n+k)p-(n-k)q}{N}\right), \quad (7)$$

where $(n+k,n-k)\in[0,\ldots,N-1]^2$.

It can be shown that it only exists one linear relationship between the columns of $\mathbf{A}$: the two columns respectively corresponding to $p=q=0$ and $p=q=N/2$ are equals, which can be verified from Eq. (7).

Since $\mathbf{A}$ is a $N^2/4+N$-1 rank matrix and H is isomorphic to $\mathcal{R}^{N^2/4+N}$, the DWV representation of an analytic signal can be generated from $N^2/4+N$-1 linearly-independent TF locations.

3.4. Example

A set of linearly-independent TF locations is represented in Figs. (1) and (2) for 16-sample analytic signals. It was experimentally determined as follows. A 2000 by 256 matrix $\mathbf{M}$ whose rows are the line-wise DWV representations of 2000 realizations of a white noise was generated. A set of d independent TF locations was found by successively analyzing the rank of nested sub-matrices made up of a part of the columns of $\mathbf{M}$. The obtained basis obviously depends on the sequence of sub-matrices used to study the linear dependencies between the columns of $\mathbf{M}$. Next, the linear relationships that allow to generate the whole DWV representation from the d independent TF locations were identified by solving N^2-d least squares problems (the errors were obviously zero). For the remainder of this paper, these relationships will be denoted by the matrix $\mathbf{B}$ which is defined as follows, after some permutations of the components of $\mathbf{W}_x$:

$$\mathbf{W}_x(i)=\sum_{j=1}^{d}\mathbf{B}(i,j)\,\mathbf{W}_x(j),\ \forall i\in\left\{d+1,\ldots,N^2\right\}, \quad (8)$$

The parameter d, that denotes the number of linearly-

Fig.1: Linearly–independant TF locations

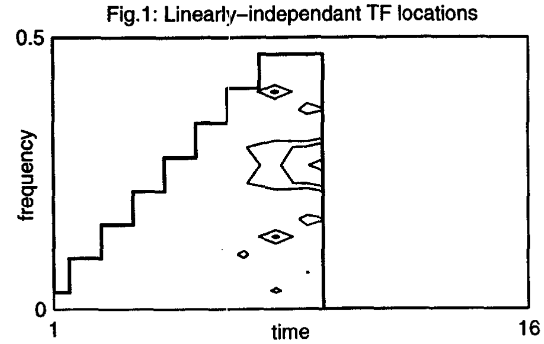
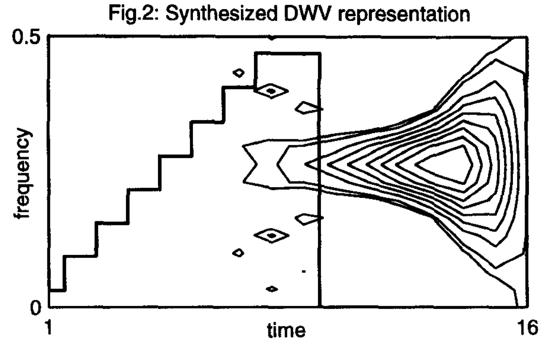


Fig.2: Synthesized DWV representation



The information conveyed by the set of linearly-independent TF locations (Fig. 1) is used to synthesized the whole DWV representation (Fig. 2). The signal is given by $x(k) = (16-k)\times\exp[-0.45\times(16-k)-2j\pi\times(16-k)/4]$, $k \in \{1, ..., 16\}$.

independent locations in the DWV representation, is equal to $N^2/4+N-1$ in Figs (1) and (2) since **x** is analytic.

4. APPLICATION TO TF DETECTORS DESIGN

It is shown in [4] that linear TF-based detectors are optimal for a wide variety of composite hypothesis testing scenarios. However, we propose to illustrate the property described in this paper with the problem (9) because the scope of our approach can easily be extended as in [5]:

$$\begin{cases} \text{if } \lambda(\mathbf{W}_x) = \mathbf{V}^T \mathbf{W}_x \geq \gamma \text{ then } H_1 \\ \text{else } H_0. \end{cases} \tag{9}$$

V is a TF reference to be determined and **x** denotes an observation.

The linear statistic $\lambda(\mathbf{W}_x)$ can be rewritten as follows:

$$\begin{aligned} \lambda(\mathbf{W}_x) &= \sum_{i=1}^{N^2} \mathbf{V}(i)\,\mathbf{W}_x(i) = \sum_{i=1}^{d} \left\{ \mathbf{V}(i) + \sum_{j=d+1}^{N^2} \mathbf{B}(j,i)\,\mathbf{V}(j) \right\} \mathbf{W}_x(i) \\ &= \sum_{i=1}^{d} \mathbf{U}(i)\,\mathbf{W}_x(i). \end{aligned} \tag{10}$$

U denotes the TF reference performing on the d linearly-independent locations, and associated with the reference **V** that exploits the information available in the whole TF domain. Reciprocally, **V** can be evaluated from **U** by solving the linear system given by:

$$\mathbf{U}(k) = \mathbf{V}(k) + \sum_{i=d+1}^{N^2} \mathbf{B}(k,i)\,\mathbf{V}(i),\ k \in \{1, \ldots, d\}, \tag{11}$$

$$\text{where } \mathbf{V}(i) = \sum_{j=1}^{d} \mathbf{B}(i,j)\,\mathbf{V}(j),\ i \in \{d+1, \ldots, N^2\}. \tag{12}$$

An experiment of blind detector design from training data was conducted in order to illustrate this approach. In the case of detecting the presence or absence of $s(k) = (16-k)\times \exp[-0.45\times(16-k)]\times\exp[-2j\pi\times((16-k)/4+\theta)]$, $k\in\{0, ..., 15\}$, in zero mean white Gaussian noise (RSB = -3 dB), with phase θ a uniform random variable, the optimal receiver is well known to correspond to the inner product of the DWV distribution $\mathbf{W}_s$ of **s** (Fig. (3)) with that of the observation **x**. The design of the TF-based detector **U** was conducted as in [6, 7] with 10000 analytic realizations of the hypotheses H_0 and H_1. The result is shown in Fig. (4). Next, we have applied Eq. (11) to evaluate $\mathbf{V}(k)$, $k\in\{0, ..., N-1\}$, (Fig. (5)) and Eq. (12) to expand the result over the whole TF domain (Fig. (6)). This reference closely resembles the DWV representation of **s**. **U** obviously yields the same performance as **V**.

5. CONCLUSION

In this paper, we have shown that the whole DWV representation of a signal x can be determined from linear combinations of some selected locations. Obviously, the Ambiguity function and the other TF representations that can be obtained from the DWV by unitary transformations also satisfy this property.

Taking this property into consideration may result in more efficient algorithms for TF-based analysis, processing, and synthesis. As an example, we have shown that it allows to efficiently compute TF-based detectors from training data, since the computation time of the training stage is nearly divided by 12 when signals are analytic. But it could also lead to new methods in various fields such as the removing of the interference terms in the DWV representation, the resolution of the signal synthesis problem, etc. However, further investigations are needed to characterize the linear relationships between the TF samples.

6. REFERENCES

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Fig. 3: DWV representation of the signal s

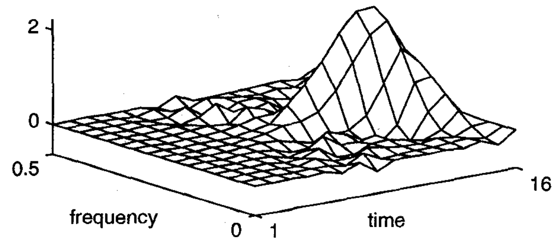


Fig. 4: TF reference U obtained from the training algorithm

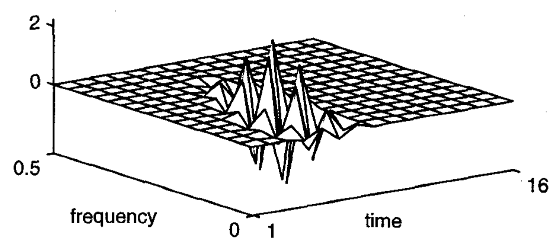


Fig. 5: TF reference V after Eq.(11) resolution

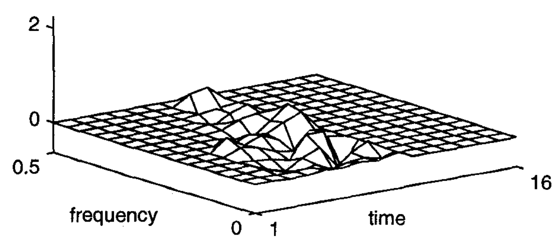
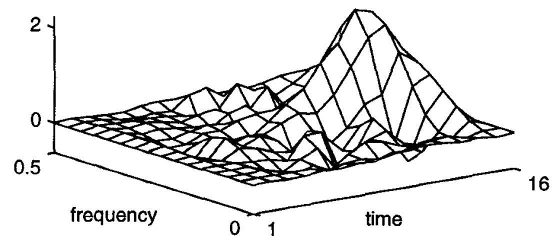


Fig. 6: Expension of V over the whole TF domain



Figs 3-6: Detection of the signal $s(k) = (16-k)\times\exp[-0.45\times(16-k)-2j\pi\times((16-k)/4+\theta)]$, $k \in [1, ..., 16]$, with phase θ a uniform random variable (Fig. 3), in a zero mean white Gaussian noise (RSB = -3 dB). The signals that are used are analytic. The detector represented in Fig. 4 only exploit the information available in a set of linearly-independent TF samples. The Fig. 6 represents the detector performing in the whole TF domain which is derived from the receiver given in Fig. 4.

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