RECURSIVE IMPLEMENTATION OF SOME TIME-FREQUENCY REPRESENTATIONS

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Abstract - Cohen's class of Time-Frequency Distributions (CTFDs), which includes the spectrogram and the Wigner-Ville distribution, has significant potential for the analysis of non-stationnary signals. In order to efficiently compute long signals time-frequency representations, we propose fast algorithms using a recursive approach. First, we introduce a recursive algorithm dedicated to the spectrogram computation. We show that rectangular, half-sine, Hamming, Hanning and Blackman functions can be used as running "short-time" windows. Then the previous algorithm is extended to specific CTFDs. We show that the windows mentioned above can also be used to compute recursively smoothed pseudo Wigner-Ville distributions. Finally, we show that the constraints on candidate windows are not very restrictive : any function (assumed periodic) can be used in practice as long as it admits a "short enough" Fourier series decomposition.

I. INTRODUCTION

Cohen's class of Time-Frequency Distributions (CTFD), which includes as particular cases the spectrogram and the Wigner-Ville distribution, has been widely used to analyse non-stationary signals [1]. In order to efficiently compute long signals time-frequency distributions, several ideas to quicken computation time have been recently proposed. Martin and Flandrin [2], Boashash and Black [3] use symmetry properties of distributions and Barry clever matrix manipulations [4]. For the evaluation of the Wigner Ville distribution, Cunningham and Williams propose a decimation algorithm which shifts the signal so that the resulting twiddle multiplication number is reduced [5]. In [6], the same authors also define approximations to real-valued CTFDs using spectrograms that admit fast evaluation. Most of the time, these algorithms are relatively ineffective for reducing computation time when applied to long signals (complex systems monitoring, biomedical signal analysis, ...).

In this paper, we propose a new approach using a recursive method, where the representation at time n is used to compute the representation at time n+1. The paper is organized as follows : first, we present a recursive algorithm for the spectrogram computation and the associated windows which can be used. Next, we propose a method dedicated to specific discrete Cohen's class distributions. Our approach is discussed in the last section.

II. RECURSIVITY FOR THE SPECTROGRAM

1. Presentation of the spectrogram recursive implementation

The spectrogram appeared in the forties under the sonagram form [1] and is still extensively used although its time and frequency resolutions are bounded. This representation is defined as :

$$S_x\left(n+\frac{N}{2},\omega\right)=\left|F_x^w(n,\omega)\right|^2 \tag{1}$$

where $F_x^w(n,\omega)=\sum_{i=1}^{N} x(n+i)\,w(i)\,e^{-j\omega i}$ (2)

In the above definition, $x(k)$ is a discrete-time complex signal, ω denotes the pulsation and w is an analysis window which plays a central role in adjusting time and frequency resolutions.

After a straightforward manipulation, eq. (2) can be written as :

$$\begin{aligned}F_x^w(n+1,\omega)=&\left[\sum_{i=1}^{N} x(n+i)\,w(i-1)\,e^{-j\omega i}\right]e^{j\omega}\\&-x(n+1)\,w(0)+x(n+N+1)\,w(N)\,e^{-j\omega N}\end{aligned} \tag{3}$$

If we consider now the case when $w(i-1) = C\,w(i)$, where C is some complex constant, (this condition is analysed below), eq. (3) becomes :

$$\begin{aligned}F_x^w(n+1,\omega)=&\,C\,F_x^w(n,\omega)\,e^{j\omega}\\&-Cx(n+1)\,w(1)+x(n+N+1)\,w(N)e^{-j\omega N}\end{aligned} \tag{4}$$

Let us analyse now the condition $w(i-1) = C\,w(i)$:

$$w(i-1)=C\,w(i)\Leftrightarrow w(i)=C^{-i}\,w(0)$$

Therefore, the W family of solutions is defined by :

$$W = \{w \,/\, w(i) = \alpha\, C^{-i},\ \alpha \text{ complex}\}$$

We can notice that W is composed of exponential functions.

2. Generalisation of the spectrogram recursive scheme

Our purpose is now to extend the family of candidate windows, i.e. allowing a recursive computation of the spectrogram. This can be done by using the linearity of the Fourier transform,

$$\text{if } w(i) = \sum_{j=1}^{J} w_j(i) \quad \text{and} \quad w_j(i-1) = C_j\, w_j(i) \qquad (5)$$

eq. (4) can be modified accordingly :

$$F_x^{w}(n+1,\omega) = \sum_{j=1}^{J} F_x^{w_j}(n+1,\omega) \qquad (6)$$

$$\begin{aligned} F_x^{w_j}(n+1,\omega) = {} & C_j F_x^{w_j}(n,\omega) e^{j\omega} \\ & - C_j\, x(n+1) w_j(1) + x(n+N+1) w_j(N)\, e^{-j\omega N} \end{aligned} \qquad (7)$$

We can notice that the expression of $F_x^{w_j}(n+1,\omega)$ makes a simple use of $F_x^{w_j}(n,\omega)$ and of an additive correction term. The global representation is obtained by the summation of the J elementary spectrograms $F_x^{w_j}(n,\omega)$.

3. Examples of suitable windows

The following windows (i.e. verifying eq. (5)) can be used for the computation of a recursive spectrogram : rectangular (J=1), half-sine (J=2), Hanning (J=3), Hamming (J=3) and Blackman (J=5). In Table 1, we introduce the way to implement rectangular and half-sine windows.

window	rectangular	half-sine
expression	$w(k) = \begin{cases} 1 \text{ for } 1 \le k \le N \\ 0 \text{ elsewhere} \end{cases}$	$w(k) = \begin{cases} \sin\left(\frac{\pi k}{N+1}\right) \text{ for } 1 \le k \le N \\ 0 \text{ elsewhere} \end{cases}$
J	1	2
w_j	$w_1(k) = 1$	$w_1(k) = \frac{1}{2j} e^{j\pi k/(N+1)}$ $w_2(k) = \frac{-1}{2j} e^{-j\pi k/(N+1)}$
C_j	$C_1 = 1$	$C_1 = \frac{1}{2j} e^{-j\pi/(N+1)}$ $C_2 = \frac{-1}{2j} e^{j\pi/(N+1)}$

Table 1 : implementation of the recursion

One can easily deduce from Table 1 the $\{w_j, C_j\}_{j=1,\dots,J}$ decomposition of Hamming, Hanning and Blackman windows, which are defined as follows :

Hanning (α=0.50) and Hamming (α=0.54) windows :

$$w(k) = \begin{cases} \alpha + (\alpha - 1)\cos\left(\frac{2\pi k}{N+1}\right) \text{ for } 1 \le k \le N \\ 0 \text{ elsewhere} \end{cases}$$

Blackman window :

$$w(k) = \begin{cases} 0.42 - 0.5\cos\left(\frac{2\pi k}{N+1}\right) + 0.08\cos\left(\frac{4\pi k}{N+1}\right) \text{ for } 1 \le k \le N \\ 0 \text{ elsewhere} \end{cases}$$

4. Evaluation of the proposed algorithm

The purpose of this section is to illustrate the computational efficiency of the proposed algorithm for several running windows. Thus we define an elementary operation (EO) as a real multiplication followed by a real addition [7]. We also consider that the total number of EO (TNEO) can be approximated by : *max* {total number of real additions, total number of real multiplications}. As a consequence, the relative efficiency (RE) of the proposed algorithm is defined as :

RE = (TNEO required to compute the representation at time n by the direct application of eq. (2)) *divided by* (TNEO required to compute the representation at time n using our algorithm)

Figure 1 represents the relative efficiency for several windows as a function of their length.

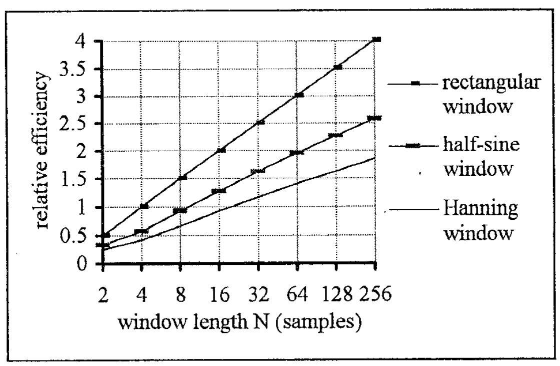

Figure 1 : relative efficiency for several windows

III. EXTENSION TO THE COHEN'S CLASS TFDs

1. Definition

The Cohen's class Time-Frequency Distributions (CTFD), which have been extensively studied in recent years, are defined as follows [1][2] :

$$CTFD_x^{\psi}(n,\omega) = \sum_{m=1-N}^{N-1} \sum_{p=-L}^{L} \psi(p,m)\, R_x(n+p,m)\, e^{-j\omega m}$$

If the kernel ψ verifies $\psi(p,-m) = \psi^*(p,m)$, it can be written after straightforward manipulations of the previous definition :

$$CTFD_x^\psi(n,\omega) = 2\,\mathrm{Re}\left[F_x^\psi(n,\omega)\right] - \sum_{p=-L}^{L} \psi(p,0)\, R_x(n+p,0) \quad (8)$$

where $F_x^\psi(n,\omega) = \sum_{m=0}^{N-1} \sum_{p=-L}^{L} \psi(p,m)\, R_x(n+p,m)\, e^{-j\omega m}$ (9)

The last definition will be used for the remainder of section III.

2. Presentation of our CTFD recursive implementation

We apply a similar strategy as in § (II.1) in order to propose a recursive implementation of CTFDs. After a straightforward manipulation of eq. (9), the CTFD can be written :

$$F_x^\psi(n+1,\omega) = \sum_{m=0}^{N-1} \sum_{p=-L+1}^{L+1} \psi(p-1,m)\, R_x(n+p,m)\, e^{-j\omega m} \quad (10)$$

Using the same assumption as in § (II.1),

if $\psi(p-1,m) = C\, \psi(p,m)$ (11)

where C is independent of m, eq. (10) becomes :

$$F_x^\psi(n+1,\omega) = C\, F_x^\psi(n,\omega) + \sum_{m=0}^{N-1} \varphi(n,m)\, e^{-j\omega m} \quad (12)$$

In the above expression, $\varphi(n, m)$ is given by :

$$\begin{aligned}\varphi(n,m) &= \psi(L,m)\, R_x(n+L+1,m)\\ &\quad - C\, \psi(-L,m)\, R_x(n-L,m)\end{aligned} \quad (13)$$

Therefore, $F_x^\psi(n+1,\omega)$ is determined using the previous result and an additive correction term, easily evaluated by a simple FFT.

3. Definition of the candidate windows family

In order to define the family G of windows which allow a recursive implementation of CTFDs, we must solve the particular functional equation (11) :

$$\psi(p-1,m) = C\, \psi(p,m) \Leftrightarrow \psi(p,m) = C^{-p}\, \psi(0,m)$$

Therefore, the family of solutions is defined by :

$$G = \left\{\psi \,/\, \psi(p,m) = C^{-p}\, f(m),\ C \text{ complex},\ f : \mathbf{Z} \to \text{Complex}\right\}$$

We can notice that G is composed of functions with separable (l,m) variables.

4. Generalisation of the CTFD recursive scheme

As in § (II.2), we use the linearity of the Fourier transform to extend the family of candidate windows:

if $\psi(p,m) = \sum_{j=1}^{J} \psi_j(p,m)$ (14)

where $\psi_j(p-1,m) = C_j\, \psi_j(p,m)$ (15)

eq. (12) becomes :

$$F_x^\psi(n+1,\omega) = \sum_{j=1}^{J} F_x^{\psi_j}(n+1,\omega) \quad (16)$$

where

$$F_x^{\psi_j}(n+1,\omega) = C_j\, F_x^{\psi_j}(n,\omega) + \sum_{m=0}^{N-1} \varphi_j(n,m)\, e^{-j\omega m} \quad (17)$$

Therefore, each elementary CTFD is computed at time n+1 using its value at time n and an additive correction term. The global representation is obtained by the summation of the J elementary spectrums.

5. Examples of candidate windows

As for the recursive implementation of the spectrogram, the following windows can be used : rectangular (J=1), half-sine (J=2), Hanning (J=3), Hamming (J=3) and Blackman (J=5) windows, considered now as functions of the variable p and post multiplied by the function f(m). One can see Table 1 to use the windows mentioned above in the CTFDs recursion.

6. Evaluation of the proposed algorithm

Figure 2 represents the relative efficiency (RE) of our algorithm, where RE is defined as in § (II.4). In this example, we chose to evaluate the distribution on 128 autocorrelation values at each time-instant.

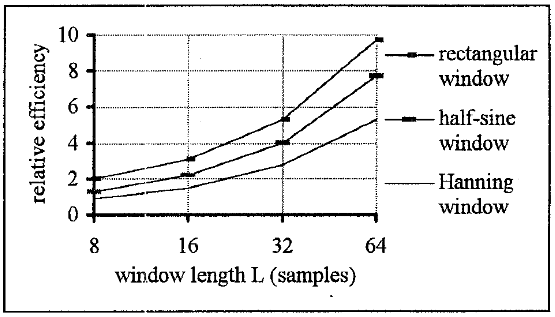

Figure 2 : relative efficiency for several windows

IV. DISCUSSION

In both TFD cases (spectrogram and CTFD), the windows used must verify an exponential decay in order to permit our recursive implementation. Consequently, if this limitation is not very restrictive in the case of the spectrogram, it only authorizes the evaluation of Cohen's class smoothed pseudo Wigner-Ville distribution. Moreover, the relative efficiency strongly depends on the number J of elementary functions used in eq. (5) and (14) (see Figure 1 and 2). Any window (assumed periodic) could thus be used as long as it admits a "short enough" Fourier series decomposition. However, the examples provided here demonstrate that the W and G window families are sufficiently rich.

We plan now to apply recursive schemes (7) and (17) to implement some reassignment methods [8][9].

V. REFERENCES

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