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FAST IMPLEMENTATION OF TIME-FREQUENCY REPRESENTATIONS MODIFIED BY THE REASSIGNMENT METHOD

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Abstract - Cohen's class Time-Frequency Distributions (CTFDs) have significant potential for the analysis of non-stationary signals, even if the poor readability of their representations makes visual interpretations difficult. To concentrate signal components, Auger and Flandrin generalized the reassignment method (first applied to the spectrogram) to any bilinear representations. Unfortunately, this process is computationally expensive. In order to quicken computation time and to improve representations readability, we first introduce a new fast algorithm which allows the recursive evaluation of classical spectrograms and spectrograms modified by the reassignment method. We show that rectangular, half-sine, Hamming, Hanning and Blackman functions can be used as running windows. Then the previous algorithm is extended to CTFDs. We show that the windows mentioned above can also be used to compute recursively reassigned smoothed pseudo Wigner-Ville distributions. Finally, we show that the constraints on candidate windows are not very restrictive : any function (assumed periodic) can be used in practice as long as it admits a "short enough" Fourier series decomposition.

I. INTRODUCTION

Cohen's class Time-Frequency Distributions (CTFD), which includes as particular cases the spectrogram and the Wigner-Ville distribution, have been widely used to analyze non-stationary signals [1]. Two major problems have been already extensively addressed : the difficulty of adjusting readability and their large computational cost.

Adjusting readability requires both a good concentration of the signal components on the time-frequency map and the absence of misleading interferences. To make visual interpretation easy, one can choose an appropriate CTFD and empirically adjust its parameters, giving an advantage either to signal concentration or interferences elimination. However, several approaches allow to automatically improve the readability of time-frequency representations. To remove cross-components, Flandrin introduces various weighting functions in computing the Wigner-Ville distribution (WVD) [2]. Sun, Li *et al.* use appropriate image processing techniques [3]. Jones and Baraniuk [4][5], Jones and Parks [6] propose efficient adaptive methods, which are computationally expensive when applied to long signals. Finally, the reassignment method, first applied 18 years ago to the spectrogram by Kodera, Gendrin and de Villedary, can produce a good localization of the signal components [7]. This method increases readability by relocating the representation values away from their location, thus creating a *reassigned CTFD*. Recently, Auger and Flandrin generalized the reassignment process, applying it to any bilinear time-frequency and time-scale representations, and simplified its implementation by proposing a new formulation [8]. Unfortunately, this method is still computationally expensive and consequently cannot be easily used in a real-time context.

In this paper, we propose a new fast algorithm which allows the evaluation of classical CTFDs and CTFDs modified by the reassignment process. This algorithm uses a recursive approach to compute both the CTFD and the reassignment stage. The paper is organized as follows : first, we present a recursive algorithm for the reassigned spectrogram computation and the associated windows which can be used. Next, we propose a method dedicated to specific discrete reassigned Cohen's class distributions. Our approach is discussed in the last section.

II. RECURSIVITY FOR THE REASSIGNED SPECTROGRAM

1. Spectrogram recursive implementation

The spectrogram appeared in the forties under the sonagram form [1] and is still extensively used although its time and frequency resolutions are bounded. This representation is defined as :

$$S_x\left(n+\frac{N}{2},\omega\right)=\left|F_x^w(n,\omega)\right|^2 \qquad (1)$$

where $$F_x^w(n,\omega)=\sum_{i=1}^{N}x(n+i)\,w(i)\,e^{-j\omega i} \qquad (2)$$

In the above definition, x(k) is a discrete-time complex signal, ω denotes the pulsation and w is an analysis window which plays a central role in adjusting time and

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frequency resolutions.

After a some manipulations, eq. (2) can be written as :

$$F_x^{w}(n+1,\omega)=\left[\sum_{i=1}^{N} x(n+i)\, w(i-1)\, e^{-j\omega i}\right] e^{j\omega} \tag{3}$$
$$-x(n+1)\, w(0)+x(n+N+1)\, w(N)\, e^{-j\omega N}$$

If we consider now the case when w(i-1) = C w(i), where C is some complex constant, (this condition is analysed below), eq. (3) becomes :

$$F_x^{w}(n+1,\omega)=C\, F_x^{w}(n,\omega)\, e^{j\omega}$$
$$-\,C x(n+1) w(1)+x(n+N+1) w(N) e^{-j\omega N} \tag{4}$$

Let us analyse now the condition w(i-1) = C w(i) :

$w(i-1)=C\, w(i) \Leftrightarrow w(i)=C^{-i}\, w(0)$

Therefore, the W family of solutions is defined by :

$$W=\left\{w \,/\, w(i)=\alpha\, C^{-i},\ \alpha \text{ complex}\right\} \tag{5}$$

We can notice that W is composed of exponential functions.

2. Recursion in the spectrogram reassignment stage

i) Problem formulation : To improve the spectrogram readability by concentrating the signal components on the time-frequency map, Kodera, Gendrin and de Villedary proposed to relocate the representation values away from their computation location [7]. The following spectrogram definition, which uses the Rihaczek distribution, is the starting point of their idea :

$$S_x(n,\omega)=\frac{1}{2\pi}\sum_{n'}\sum_{\omega'} Ri^{*}(n',\omega';w^{*})\, Ri(n-n',\omega-\omega';x)$$

where $Ri(n,\omega;x)=x(n)\, X^{*}(\omega)\, e^{-jn\omega}$

The above definition shows that the spectrogram at any location (n, ω) can be considered as the average of the weighted Rihaczek distribution values at the neighboring points (n − n', ω − ω'). Because this smoothing leads to a signal components broadening, the authors suggested to change the attribution point of the average, assigning it to center of gravity $(\hat{n},\hat{\omega})$ of the weighted distribution. This process yields a reassigned spectrogram whose value at any location is therefore the sum of all representation values relocated to this point. Recently, Auger and Flandrin [8] simplified the reassignment algorithm by proposing the following expressions to compute new locations as a function of the time-frequency coordinates (n, ω) :

$$\hat{n}(n,\omega;x)=n-\frac{N}{2}+\Re e\left\{\frac{F_x^{Tw}(n,\omega)\, F_x^{w}(n,\omega)^{*}}{\left|F_x^{w}(n,\omega)\right|^{2}}\right\} \tag{6}$$

$$\hat{\omega}(n,\omega;x)=\omega-\Im m\left\{\frac{F_x^{Dw}(n,\omega)\, F_x^{w}(n,\omega)^{*}}{\left|F_x^{w}(n,\omega)\right|^{2}}\right\} \tag{7}$$

where $F_x^{Tw}(n,\omega)=\sum_{i=1}^{N} x(n+i)\, i\, w(i)\, e^{-j\omega i}$ (8)

and $F_x^{Dw}(n,\omega)=\sum_{i=1}^{N} x(n+i)\, w'(i)\, e^{-j\omega i}$ (9)

One can notice that $F_x^{W}(n,\omega)$ recursively evaluated during the spectrogram computation, is also used in the reassignment process.

ii) Recursivity of the reassignment operators : When using Auger and Flandrin's formulation, the reassignment process requires the evaluation of two additional FFT at each time-instant n, which can be computationally expensive. This justifies the search for recursive expressions of eq. (8) and (9).

After a straightforward manipulation, eq. (8) can be written as :

$$F_x^{Tw}(n+1,\omega)=\left[\sum_{i=1}^{N}[i-1]x(n+i)w(i-1)\, e^{-j\omega i}\right]e^{j\omega} \tag{10}$$
$$+N x(n+N+1) w(N)\, e^{-j\omega N}$$

If we consider now the case when w belongs to the W family, eq. (10) becomes :

$$F_x^{Tw}(n+1,\omega)=C\left[F_x^{Tw}(n,\omega)-F_x^{w}(n,\omega)\right]e^{j\omega} \tag{11}$$
$$+N x(n+N+1) w(N)\, e^{-j\omega N}$$

Therefore, we can notice that $F_x^{Tw}(n+1,\omega)$ is easily determined by using $F_x^{Tw}(n,\omega)$, $F_x^{w}(n,\omega)$ and an additive correction term.

Let us simplify now eq. (9) when w belongs to W :

$w(i)=\alpha\, C^{-i} \Rightarrow w'(i)=-\,\alpha\,(\ln C)\, C^{-i}$

As a consequence, $F_x^{Dw}(n+1,\omega)$ computation requires only one complex multiplication :

$$F_x^{Dw}(n+1,\omega)=-(\ln C)\, F_x^{w}(n+1,\omega) \tag{12}$$

3. Generalization of the recursive scheme

Our purpose is now to extend the family of candidate windows, i.e. allowing a recursive computation of the reassigned spectrogram. This can be done by using the linearity of the Fourier transform,

if $w(i)=\sum_{j=1}^{J} w_j(i)$ and $w_j(i-1)=C_j\, w_j(i)$ (13)

eq. (4), (11) and (12) can be modified accordingly :

$$F_x^{\chi}(n+1,\omega)=\sum_{j=1}^{J} F_x^{\chi_j}(n+1,\omega) \tag{14}$$

where χ symbolizes the following running windows : w, Dw and Tw (see def. (2), (8) and (9)). Using the same notation, $F_x^{\chi_j}$ (j = 1, ..., J) are recursively evaluated according to eq. (4), (11) and (12).

Therefore, the W family can be extended as follows :

$$W = \left\{ w \,/\, w(i) = \sum_{j=1}^{J} \alpha_j \, C_j^{-i}, \alpha_j \text{ complex} \right\} \qquad (15)$$

We can notice that w is a linear combination of complex exponential functions. Consequently, the following windows can be used for the recursive computation of a reassigned spectrogram : rectangular (J=1), half-sine (J=2), Hanning (J=3), Hamming (J=3) and Blackman (J=5) windows.

4. Evaluation of the proposed algorithm

The purpose of this section is to illustrate the computational efficiency of the proposed algorithm for several running windows. Thus we define an elementary operation (EO) as a real multiplication followed by a real addition [9]. We also consider that the total number of EO (TNEO) can be approximated by : *max* {total number of real additions, total number of real multiplications}. As a consequence, the relative efficiency (RE) of the proposed algorithm is defined as :

RE = (TNEO required to compute the representation at time n by the direct application of eq. (2), (8) and (9)) *divided by* (TNEO required to compute the representation at time n using our algorithm)

Figure 1 represents the recursive algorithm relative efficiency for several windows as a function of their length.

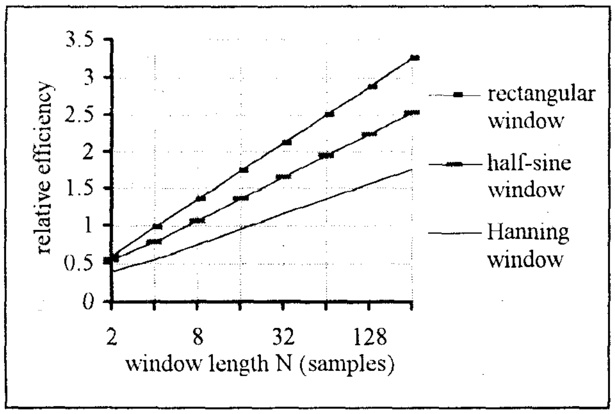

Figure 1 : RE of the reassigned spectrogram algorithm

III. EXTENSION TO THE REASSIGNED COHEN'S CLASS TFDs

1. Definition

The Cohen's class Time-Frequency Distributions, which have been extensively studied in recent years, are defined as follows [1] :

$$TF_x^{\psi}(n,\omega) = \sum_{m=1-N}^{N-1} \sum_{p=-L}^{L} \psi(p,m)\, R_x(n+p,m)\, e^{-j2\omega m} \qquad (16)$$

where R_x represents the instantaneous autocorrelation function of the studied signal and ψ the distribution smoothing kernel.

2. CTFD recursive implementation

We apply a similar strategy as in § (II.1) to propose a recursive implementation of CTFDs. After a straightforward calculus, the CTFD can be written as :

$$TF_x^{\psi}(n+1,\omega) = \sum_{m=1-N}^{N-1} \sum_{p=-L+1}^{L+1} \psi(p-1,m)\, R_x(n+p,m)\, e^{-j2\omega m} \qquad (17)$$

Using the same assumption as in § (II.1),

if $\psi(p\text{-}1,m) = C\ \psi(p,m)$ (18)

where C is independent of m, eq. (17) becomes :

$$TF_x^{\psi}(n+1,\omega) = C\ TF_x^{\psi}(n,\omega) + \sum_{m=1-N}^{N-1} \varphi(n,m)\, e^{-j2\omega m} \qquad (19)$$

In the above expression, $\varphi(n, m)$ is given by :

$$\begin{aligned}\varphi(n,m) &= \psi(L,m)\, R_x(n+L+1,m) \\ &\quad - C\ \psi(-L,m)\, R_x(n-L,m)\end{aligned} \qquad (20)$$

Therefore, $TF_x^{\psi}(n+1,\omega)$ is determined using its previous value and an additive correction term, easily evaluated by a (2N-1)-sample FFT.

Let us solve now the functional equation (18) :

$\psi(p-1,m) = C\ \psi(p,m) \Leftrightarrow \psi(p,m) = C^{-p}\ \psi(0,m)$

Therefore, the family G of solutions is defined by :

$$G = \left\{ \psi \,/\, \psi(p,m) = g(p)\ h(m) \text{ with } g(p) = C^{-p},\ C \text{ complex} \right\}$$

We can notice that G is composed of functions with separable (p,m) variables. Consequently, our algorithm only authorizes the evaluation of smoothed pseudo Wigner-Ville distributions (SPWVD).

3. Reassignment stage recursive implementation

i) Problem formulation : In [8], Auger and Flandrin showed that the reassignment method can be advantageously applied to any CTFD. In the SPWVD case, it can be implemented as follows :

$$\hat{n}(n,\omega;x) = n - \frac{TF_x^{Tgh}(n,\omega)}{TF_x^{gh}(n,\omega)} \qquad (21)$$

$$\hat{\omega}(n,\omega;x) = \omega + j\frac{TF_x^{gDh}(n,\omega)}{TF_x^{gh}(n,\omega)} \qquad (22)$$

where gh (p, m) = g(p) h(m),

$$TF_x^{gDh}(n,\omega) = \sum_{m=1-N}^{N-1} \sum_{p=-L}^{L} g(p)\, h'(m)\, R_x(n+p,m)\, e^{-j2\omega m} \qquad (23)$$

and

$$TF_x^{Tgh}(n,\omega) = \sum_{m=1-N}^{N-1} \sum_{p=-L}^{L} p\, g(p)\, h(m)\, R_x(n+p,m)\, e^{-j2\omega m} \qquad (24)$$

One can notice that TF_x^{gh} is recursively evaluated during the SPWVD computation (see eq. (19)).

ii) Reassignment stage recursivity : Our purpose is now to propose an efficient recursive implementation of eq. (23) and (24). We will consider for the remainder of this section that the kernel gh belongs to the family G.

After a straightforward manipulation of eq. (23), we can write :

$$TF_x^{gDh}(n+1,\omega) = C\,TF_x^{gDh}(n,\omega) + \sum_{m=1-N}^{N-1} \eta(n,m)\,e^{-j2\omega m} \quad (25)$$

In the above expression, $\eta(n, m)$ is given by :

$$\begin{aligned}\eta(n,m) = \big[&g(L)\,R_x(n+L+1,m) \\ &- C\,g(-L)\,R_x(n-L,m)\big]\,h'(m)\end{aligned} \quad (26)$$

Using the same strategy as in eq. (11), eq. (24) can be written after some calculus:

$$TF_x^{Tgh}(n+1,\omega) = C\left[TF_x^{Tgh}(n,\omega) - TF_x^{gh}(n,\omega)\right] + \sum_{m=1-N}^{N-1} \mu(n,m)\,e^{-j2\omega m}$$

where

$$\begin{aligned}\mu(n,m) = \big[&L\,g(L)\,R_x(n+L+1,m) \\ &+ C\,(L+1)\,g(-L)\,R_x(n-L,m)\big]\,h(m)\end{aligned} \quad (27)$$

One can notice that TF_x^{gDh} and TF_x^{Tgh} make a simple use of their previous values and of additive correction terms evaluated by means of a FFT.

4. Generalization of the recursive scheme

As in § (II.3), we can use the linearity of the Fourier transform to extend the family of suitable windows :

$$G = \left\{\psi \,/\, \psi(p,m) = h(m) \sum_{j=1}^{J} C_j^{-p},\ C_j \text{ complex}\right\} \quad (28)$$

Consequently, the following windows allow a recursive implementation of SPWVD : rectangular (J=1), half-sine (J=2), Hanning (J=3), Hamming (J=3) and Blackman (J=5), considered now as functions of the variable p and post multiplied by the function f(m).

5. Evaluation of the proposed algorithm

Figure 2 represents the relative efficiency (RE) of our algorithm, where RE is defined as in § (II.4). In this example, h (m) is a 128-sample window.

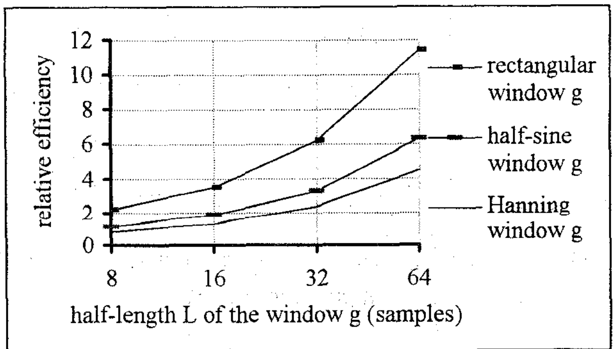

Figure 2 : relative efficiency for several windows

IV. DISCUSSION

In both reassigned TFD cases (spectrogram and CTFD), the windows used must verify an exponential decay in order to permit our recursive implementation. Consequently, if this limitation is not very restrictive in the case of the spectrogram, it only authorizes the evaluation of Cohen's class smoothed pseudo Wigner-Ville distribution. Moreover, the relative efficiency strongly depends on the number J of elementary functions used in eq. (15) and (28). Any window (assumed periodic) could thus be used as long as it admits an "short enough" Fourier series decomposition. However, the examples provided here demonstrate that the W and G window families are sufficiently rich.

V. REFERENCES

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