Reassignment of diffused time-frequency distributions

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Abstract

In this paper, we propose to combine diffusion methods [1], [2], [3] and reassignment techniques [4], [5] for the processing of time-frequency distributions. We briefly recall both techniques and propose a reassigned adaptive diffused distribution. We provide a criteria for the choice of τ, the amount of diffusion applied prior to reassignment, and illustrate with a comparison between a reassigned spectrogram and a reassigned diffused distribution. Both distributions figure high localization properties but the later possess more regularity.

I. Introduction

Because time-frequency representations (TFR) illustrate evolutions of signals with respect to both time and frequency, they have been largely used to deal with non-stationary environment. Among the host of solutions that have been proposed, Cohen class encloses bilinear TFR that are covariant with respect to time shifts and frequency shifts. Such tools lead to sharper representation of a signal than linear-based approaches, e.g., spectrograms, but at the cost of undesirable cross-terms [6]. One main goal of time-frequency smoothing is to improve readability by removing these cumbersome cross-terms while preserving the sharpness of signal terms.

In the context of homogeneous smoothing, a low-passed kernel is chosen such that the trade-off between readability and sharpness of signal terms is maximized. The radial gaussian kernel proposed in [7] is such a kernel. Noting that the processed signal is non-stationary, many authors proposed to use locally adaptive techniques. In this paper we deal with two of them, namely adaptive diffusion [1], [2], [3] and distribution reassignment [4], [5]. We propose to combine these two techniques to obtain reassigned diffused distributions. These distributions meld the sharpness of reassigned ones with the regularity of diffused representations.

II. Time-frequency distribution diffusion

In this first part we briefly recall the principles of adapted diffusion. For a more thorough treatment, the reader shall refer to [1], [2], [3]. We start by a presentation of homogeneous diffusion. We then sketch the link between diffused diffusions and spectrograms. In a last part we review adaptive diffusion.

A. Homogeneous diffusion

Diffusion is the process by which matter is transported from areas of high concentration to areas of lower concentration as a result of the movement of an ensemble of molecules inside a region. This is mathematically formulated by a basic equation referred to as Fick's law [8]:

$$J = -C\,\nabla U, \tag{1}$$

where J is the flux of molecules, U their concentration, C the diffusion tensor, and ∇ the gradient operator. The negative sign indicates that the diffusing mass flows in the direction of decreasing concentration. If C is a scalar-valued conductance function, which implies that J and ∇U are collinear, the diffusion process is called *isotropic*. Otherwise it is called *anisotropic*. It is said that the diffusion process is *homogeneous* if the diffusion tensor C is constant over the region of interest. Location-dependent diffusion is called *non-homogeneous* or *inhomogeneous*. The law of conservation of mass is expressed as:

$$\frac{\partial U}{\partial \tau} = -\mathrm{div}(J)\,, \tag{2}$$

where τ is the diffusion time, and div the divergence operator. Combining this relationship with Fick's law produces the law of diffusion, which states:

$$\frac{\partial U}{\partial \tau} = -\mathrm{div}(-C\,\nabla U)\,. \tag{3}$$

In the case where C is a positive constant, the resulting isotropic and homogeneous process (3) is often referred to as *heat diffusion equation* since it describes the evolution of temperature within a finite homogeneous continuum with no internal sources of heat. In this paper, this diffusion is said to be *linear* because the tensor C does not vary with τ. Otherwise, it would be called *nonlinear*[1]. Let us now restrict our discussion to the partial differential equations (PDE's) in two-plus-one dimension

$$\left\{ \begin{array}{l} U(v_1, v_2; \tau = 0) = U_0(v_1, v_2) \\ \frac{\partial U}{\partial \tau} = \operatorname{div}(\nabla U)\,, \end{array} \right. \tag{4}$$

where $U_0 \in \mathcal{L}^1(\mathbb{R}^2)$ denotes the initial spatial condition. It is well-known that the solution to (4) is

$$U(v_1, v_2; \tau) = \left\{ \begin{array}{ll} (G * U_0)(v_1, v_2) & (\tau > 0) \\ U_0(v_1, v_2) & (\tau = 0), \end{array} \right. \tag{5}$$

with $*$ the usual 2-D convolution, and $G(v_1, v_2; \tau) = (4\pi\tau)^{-1} \exp(-[v_1^2 + v_2^2]/4\tau)$ an isotropic Gaussian kernel which is referred to as Green's function of the PDE given above. This means that the solution $U(v_1, v_2; \tau)$ of the heat diffusion equation (4) at each time instant τ can be simply obtained by convolution of the initial spatial condition $U_0(v_1, v_2)$ with Green's function $G(v_1, v_2; \tau)$.

Among Cohen class, the spectrogram is a widely used tool. As the square modulus of the short-time Fourier transform, it can also be written as a convolution between the Wigner distribution of the signal and that of the analyzing window. Note that the Wigner distribution of a gaussian window is a 2D-gaussian kernel. Interpreting time-frequency representation as a heat distribution one can consider its diffusion as follows:

$$\left\{ \begin{array}{lcl} D_x(t, f; \tau = 0) & = & W_x(t, f) \\ \frac{\partial D_x(t,f;\tau)}{\partial \tau} & = & \operatorname{div}_{t,f}(\nabla_{t,f} D_x(t, f; \tau)), \end{array} \right. \tag{6}$$

where W_x is the representation to be processed, which plays the role of the initial state of the diffusion process. The diffused representation $D_x(t, f; \tau)$ denotes the energy distribution at the time instant τ. As we just stated, the fundamental solution of such classical heat diffusion is an isotropic gaussian. Therefore the partial derivative equation (6) has the following solution:

$$D_x(t, f; \tau) = \frac{1}{4\pi\tau} \iint W_x(\eta, \nu) e^{-\frac{(t-\eta)^2 + (f-\nu)^2}{4\tau}} d\eta d\nu \tag{7}$$

Indeed the use of the heat diffusion on a Wigner distribution is equivalent to convolving it with a gaussian kernel whose variance increases with the diffusion time τ. Note that the convolution form of the solution ensures the preservation of covariance with respect to time and frequency shifts.

B. Adaptive diffusion

In a time-frequency distribution smoothing context, adaptive diffusion was introduced in [1]. In this paper, authors propose to locally tune the diffusion process, allowing a locally adapted smoothing. Such a scheme reads:

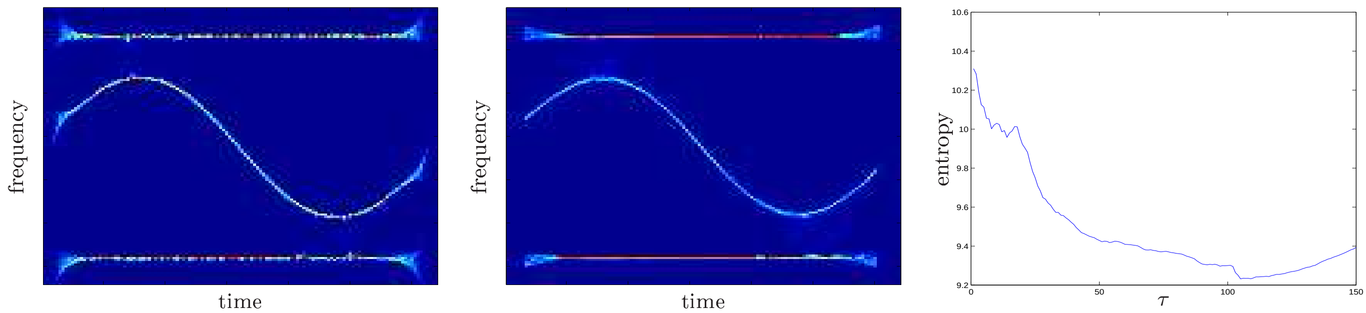
$$\left\{ \begin{array}{lcl} D_x(t, f; \tau = 0) & = & W_x(t, f) \\ \frac{\partial D_x(t,f;\tau)}{\partial \tau} & = & \operatorname{div}_{t,f}(c_x(t, f) \nabla_{t,f} D_x(t, f; \tau)), \end{array} \right. \tag{8}$$

where W_x is the to-be-smoothed distribution, D_x is the smoothed distribution up to time τ and c_x is the function controlling locally the amount of smoothing. This later is referred to as the conductance function.

In an analysis context, the aim is to remove cross terms while preserving the sharpness of signal terms. In [1], authors proposed to scalar valued function for such a goal, this diffusion is called *isotropic*. This technique relies on the spectrogram, which is approximately equal to zero over non-energetic areas where the interference terms of the WD are likely to be situated. Thus making the conductance function a decreasing function of S_x such as

$$c_{S_x}(t, f) = \left[1 + \left(\frac{SP_x(t, f)}{\beta} \right)^{\alpha} \right]^{-1} (\alpha, \beta) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \tag{9}$$

allows for the tuning of the diffusion rate over the time-frequency domain, see [2], [3] for extensions. In a nutshell, diffusion methods start from a concentrated distribution with cross terms and selectively smooth them out with a maximum preservation of sharpness for signal terms.



(a) Reassigned spectrogram (b) Reassigned diffused distribution (c) Evolution of entropy

Fig. 1. Comparison between (a) the reassignment of a spectrogram and (b) the reassignment of a distribution processed by adaptive diffusion (isotropic in that case). Fig-(c) illustrates the evolution of the entropy, with $\alpha = 3$, of the reassigned diffusion as a function of τ. Chosen stopping time is here $\tau^* = 105$.

III. Time-frequency distribution reassignment

Another very popular method tackles the problem from another perspective. Reassignment method starts from an interference free distribution, like a spectrogram for example, and increases its sharpness. This enables to obtain both sharpness and readability as the starting distribution just lacks resolution. The interference-free distribution results from the low-pass filtering of a Wigner distribution. This has two consequences: elimination of interferences and delocalization of signal-terms. Knowing the window used, reassignment relocalizes the later. Indeed for a spectrogram S_x using a window $h(t)$:

$$S_x(t, f; h) = \iint W_x(\eta, \nu) W_h(\eta - t, \nu - f) d\eta d\nu, \tag{10}$$

reassignment reads [4], [5]:

$$\hat{S}_x(t, f; h) = \iint S(\eta, \nu; h) \delta(t - \hat{t}_x^h(\eta, \nu), f - \hat{f}_x^h(\eta, \nu)) d\eta d\nu, \tag{11}$$

where $\hat{t}_x$ and $\hat{f}_x^h$ indicates the locus where the energy has to be moved to. They are given by

$$\hat{t}_x^h(t, f) = \iint \eta W_x(\eta, \nu) \frac{W_h(\eta - t, \nu - f)}{S_x(t, f; h)} d\eta d\nu \quad \text{and} \quad \hat{f}_x^h(t, f) = \iint \nu W_x(\eta, \nu) \frac{W_h(\eta - t, \nu - f)}{S_x(t, f; h)} d\eta d\nu.$$

As all the members of the Cohen class can be written in a convolutive way, they all posses reassigned versions. One could use the equivalence of homogeneous diffusion with gaussian spectrograms and smoothed pseudo Wigner distributions to proposed reassigned homogenous diffused distributions. However this relies on the Green function of the diffusion which is not available in the case of adaptive diffusion. We therefore propose another technique.

IV. Diffused distribution reassignment

Recall that there is an equivalence between the distribution D_x, result from a diffusion up to time τ according to:

$$\begin{cases} D_x(t, f; \tau = 0) &= W_x(t, f) \\ \frac{\partial D_x(t, f; \tau)}{\partial \tau} &= \operatorname{div}_{t,f}(c_x \nabla_{t,f} D_x(t, f; \tau)), \end{cases} \tag{12}$$

with $c_x(t, f)$ a constant function and the convolution

$$D_x(t, f; \tau) = \iint c_x G(\eta, \nu; \tau) W_x(\eta - t, \nu - f) d\eta d\nu. \tag{13}$$

The kernel G is the Green function of the aforementioned diffusion. In this case, that kernel G is a 2D gaussian kernel of variance $\sigma = \sqrt{2\tau}$.

In the adaptive case, we assume the Green function exists and we note it $G_{(x,\tau,t,f)}$. Because this is equivalent to use either the convolution with the Green function or the diffusion, we propose to determine the locus of the reassignment

[1]In the theory of partial differential equations, a diffusion process is called *nonlinear* if C is a function of U or its derivatives.

points using:

$$\begin{cases} \bar{t}_x(t,f;\tau=0) &= tW_x(t,f) \\ \frac{\partial \bar{t}_x(t,f;\tau)}{\partial \tau} &= \mathrm{div}_{t,f}(c_x(t,f)\nabla_{t,f}\bar{t}_x(t,f;\tau)), \end{cases} \quad \text{and} \quad \begin{cases} \overline{f}_x(t,f;\tau=0) &= fW_x(t,f) \\ \frac{\partial \overline{f}_x(t,f;\tau)}{\partial \tau} &= \mathrm{div}_{t,f}(c_x(t,f)\nabla_{t,f}\overline{f}_x(t,f;\tau)). \end{cases} \tag{14}$$

Reassignment reads then:

$$\hat{D}_x(t,f;\tau) = \iint D_x(\eta,\nu;\tau)\delta\left(t - \frac{\bar{t}_x(\eta,\nu;\tau)}{D_x(t,f;\tau)}, f - \frac{\overline{f}_x(\eta,\nu;\tau)}{D_x(t,f;\tau)}\right) d\eta d\nu. \tag{15}$$

Without constraints on the diffusion time, D_x would converge to a uniform distribution over the time-frequency domain regardless of the analyzed signal. Hence, this reassigned diffusion process is to be stopped when some criterion is achieved. The Rényi entropy, defined as [9]

$$H_{\hat{D}}(\tau) = -\frac{1}{1-\alpha}\log \iint \hat{D}_x^{\alpha}(t,f;\tau)dt\ df,$$

is a natural candidate for measuring the concentration of TFRs[2]. While in previous studies, e.g. [3], [2], we used the entropy of the diffused distribution as a stopping criteria, we here propose to use the entropy of the reassigned distribution, as it is the distribution we are interested in. While the first iterations of the diffusion are smoothing out interference terms, the later ones slowly regularize the signal terms until some point where even reassignment cannot produce a sharp distribution. This yield a minimum in the entropy curve that we use as a stopping point for the diffusion process, see fig.-1 (c) for an illustration.

As we can see on the fig.-1(a-b), both representations are interference free and very concentrated. We note that using a diffused distribution yield a more regular result combining the good properties of both techniques. This denotes a smaller entropy. Indeed note that the entropy of the figured reassigned spectrogram is 9.4 whereas the entropy of the reassigned diffused distribution is 9.2.

V. Conclusion

We have proposed an extension of the reassignment technique to tackle the diffused distributions. While demonstrated on an isotropic adaptive time-frequency distribution, it is easily extensible to include the recent developments of diffusion processing (anisotropy and processing of time-scale distributions). We then illustrated the gain over a reassigned spectrogram by exploiting the time of diffusion τ to obtain a regular, interference-free and well localized distribution.

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[2] We suggest the reader to refer to [9] for detailed study of the Rényi's entropies as time-frequency information measures. We also invite him to consult [10], [11] for full details on the Rényi's entropies as a means of extracting information from images during diffusion.