DIFFUSION EQUATIONS FOR ADAPTIVE AFFINE DISTRIBUTIONS

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ABSTRACT

In this paper, we propose an extension of the adaptive diffusion technique for time-frequency representations proposed by Payot and Gonçalvès in 1998. Instead of processing time-frequency representations and keeping the covariance with respect to time and frequency shifts untouched, our adaptive filtering technique processes time-scale representations of the affine class while preserving the covariance properties of such representations. In order to obtain representations with improved readability, we aim at removing cumbersome interference terms while not blurring the signal terms. We show that the association of a conductance function to our diffusion scheme can make significant improvement toward reaching this goal. Indeed a conductance function provides a way to adapt locally the amount of smoothing to the representation. Note that the adaptivity of this affine technique is not based on any waveform dictionary as matching pursuit algorithms.

1. INTRODUCTION

Depending on the analyzed signal and on the application in sight, one must choose an appropriate representation. In time-frequency/time-scale analysis, representations are grouped by their ability to reflect the application of a displacement operator on the signal. For example, the Cohen class encloses time-frequency representations that are covariant with respect to time shifts and frequency modulations while the affine class consists of representations covariant with respect to time shifts and dilations [1]. In this paper, we deal with bilinear representations of the affine class. We present a way to adaptively and iteratively smooth affine representations, while preserving its covariance properties. It consists of a diffusion-based technique [2, 3] extending the work presented in [4] for the Cohen class of TF representations. Indeed, our scheme is an alternative to the reassignment method [5] and to methods based on decomposition of signals on waveform dictionaries [6] that also preserve covariance properties with respect to time shifts and dilations. Since it is an iterative approach, it enables to control the amount of processing of the representation while its adaptivity provides an action that is locally adapted to the analyzed signal.

In a first part we briefly present the affine class and the covariance properties of representations that belong to it. In a second part we describe the adaptive smoothing technique presented by Gonçalvès and Payot in [4]. As it does not preserve the affine covariance properties, we present an evolution of it that preserves membership of processed representations to the affine class. Following [4], we next propose to locally adapt smoothing to the analyzed signal using a conductance function, and we show how it can be used to improve the readability of affine class representations.

2. AFFINE OPERATOR AND AFFINE CLASS

In order to clarify the problem at hand we first review the concepts of operators and covariance with respect to these operators. The time shift and frequency modulation operators underlie the Cohen class. In other words this class encompasses all the distributions that reflect the application of these operators to the signal. Similarily, the affine class is based on the affine operator [1], here denoted as $\mathcal{A}$, whose action on the set $L^2(\mathbb{R})$ is as follows:

$$\mathcal{A}x(t) = |a_0|^{-1/2} x\left(\frac{t-t_0}{a_0}\right), \tag{1}$$

where t_0 is the amount of time shifting and a_0 the amount of dilatation of the signal. A time-frequency representation Ω_x is covariant with respect to this operator if it obeys the following relation:

$$\Omega_{\mathcal{A}x}(t,f) = \Omega_x\left(\frac{t-t_0}{a_0}, a_0.f\right). \tag{2}$$

As an example, the well-known Wigner distribution,

$$W_x(t,f) = \int_{-\infty}^{\infty} x\left(t+\frac{\tau}{2}\right) x^*\left(t-\frac{\tau}{2}\right) e^{-i2\pi f\tau} d\tau,$$

is covariant with respect to the affine operator. As shown in [1], any members of the affine class can be expressed as an affine convolution of a Wigner distribution W_x with a signal-independent kernel Π as follows

$$\Omega_x(t,a) = \int_{-\infty}^{\infty}\int_{0}^{\infty} \Pi\left(\frac{s-t}{a}, a\xi\right) W_x(s,\xi)\, ds\, d\xi. \quad (3)$$

Note that conventionally, in many cases we can use the $a = f_0/f$ equivalence, with f_0 a constant, to relate frequency and scale [1].

3. SMOOTHING VIA DIFFUSION

Bilinear time-frequency or time-scale distributions allow for a sharper representation of a signal than linear-based approaches (e.g. scalogram), but a the cost of undesirable cross terms inherent to their quadratic form. One main goal of time-frequency or time-scale smoothing is to improve this readability by removing the cumbersome cross terms while preserving the sharpness of signal terms. Because of the oscillating nature of interference terms, a usual scheme consists in using low pass kernels Π.

3.1. Diffusion for the Wheyl Group

Homogenous diffusion

Among Cohen's class, the spectrogram is a widely used tool. Square modulus of the short time Fourier transform, it can also be written as a convolution between the Wigner distribution of the signal and the Wigner distribution of the analyzing window. Note that the Wigner distribution of a gaussian window is a gaussian kernel.
In [4], authors remind that the gaussian kernel is the Green function of the heat diffusion. Indeed the use of the heat diffusion on a Wigner distribution is equivalent to convolving it with a gaussian kernel whose variance depends on the diffusion time τ. Authors then suggest to use an iterative diffusion process inspired by [2] to smooth bilinear time-frequency representations. Such a scheme reads

$$\begin{cases} D_x(t,f;\tau=0) \;=\; W_x(t,f) \\ \frac{\partial D_x(t,f;\tau)}{\partial \tau} \;=\; \mathrm{div}_{t,f}(\nabla_{t,f} D_x(t,f;\tau)), \end{cases} \quad (4)$$

where W_x is the representation to be processed, here the Wigner distribution, and $D_x(t,f;\tau)$ the smoothed representation at the time instant τ. This diffusion is called homogeneous. At a certain diffusion time τ, a representation smoothed via this diffusion is equivalent to a gaussian window spectrogram.

Adaptive diffusion

As we deal with non-stationary signals, we need to locally adapt the diffusion process to the signal. Following the idea of Perona and Malik in [2], Payot and Gonçalvès propose to use a conductance function $c_x(t,f)$ to locally control the action of the diffusion. Adaptive diffusion can be written as follows

$$\begin{cases} D_x(t,f;\tau=0) \;=\; W_x(t,f) \\ \frac{\partial D_x(t,f;\tau)}{\partial \tau} \;=\; \mathrm{div}_{t,f}(c_x(t,f)\nabla_{t,f} D_x(t,f;\tau)). \end{cases} \quad (5)$$

The choice of the conductance function depends on the application on sight and on available a priori information. In a context of signal analysis, it can be used to selectively smooth cross terms while preserving signal terms [4]. For the use of diffusion in a decision making context, one should refer to [7].

3.2. Covariant affine diffusions

Homogeneous affine smoothing

As covariance with respect to time shifts and dilations is central for representations of the affine class, we shall now propose a diffusion scheme preserving this property.

Similarly with the spectrogram within the Cohen class, the scalogram plays a preponderant role within the affine class. It is the square modulus of a continuous wavelet transform. Using a gaussian wavelet, one can interpret it as the convolution in time of the analyzed signal with a gaussian window whose width is increasing with the scale of analysis, whereas it is *affine convolved* in frequency. Such an adaptation of the width of the kernel with the scale of analysis can also be observed in (3). We then propose to adapt the diffusion strength with the scale as follows [1],

$$\begin{cases} A_x(t,a;\tau=0) \;=\; \Omega_x(t,a) \\ \frac{\partial A_x(t,a;\tau)}{\partial \tau} \;=\; \mathrm{div}_{t,a}(a^2\nabla_{t,a} A_x(t,a;\tau)), \end{cases} \quad (6)$$

where A_x is the time-scale smoothed representation and Ω_x the representation to be processed. This diffusion will be called homogeneous as it does not adapt to the analyzed signal.
Due to the linear nature of the differential operators involved in this diffusion scheme, the resulting distribution remains bilinear. In addition, it is covariant with respect to affine operator as it si shown below. As stated in [1], such a distribution belongs to the affine class and therefore can be written as an affine convolution of the type (3) with some

[1] This diffusion acts on time-scale distributions. The construction of an affine diffusion for time-frequency representations is possible but requires different treatments. One can use an affine tensor instead of the scalar term a^2 as presented in [3] for image processing.

specific kernel corresponding to the Green solution of the diffusion equation (6).

We now prove that it preserves the affine covariance property. It is clear that, at diffusion step $\tau = 0$, the smoothed representation is covariant with the affine operator. Therefore, we just have to verify that the divergence term is also covariant to prove that $A(t, a; \tau)$ belongs to the affine class for any positive τ. The diffusion term can be developed as follows:

$$\begin{aligned}&\frac{\partial A_x(t,a;\tau)}{\partial \tau}\\ &= \mathrm{div}_{t,a}(a^2 \nabla_{t,a} A_x(t,a;\tau))\\ &= \mathrm{div}_{t,a}\left(a^2\left[\frac{\partial A_x}{\partial t}(t,a).\vec{u_t} + \frac{\partial A_x}{\partial a}(t,a).\vec{u_a}\right]\right)\\ &= a^2\frac{\partial^2 A_x}{\partial t^2}(t,a) + a^2\frac{\partial^2 A_x}{\partial a^2}(t,a) + 2a\,\frac{\partial A_x}{\partial a}(t,a).\end{aligned}$$

For a shifted and scaled signal $\mathcal{A}x(t)$, the diffusion reads

$$\begin{aligned}&\frac{\partial A_{\mathcal{A}x}(t,a;\tau)}{\partial \tau}\\ &= \mathrm{div}_{t,a}\left(a^2\nabla_{t,a}A_x\left(\frac{t-t_0}{a_0}, \frac{a}{a_0}; \tau\right)\right)\\ &= \mathrm{div}_{t,a}\left(\frac{a^2}{a_0}\frac{\partial A_x}{\partial t}\left(\frac{t-t_0}{a_0}, \frac{a}{a_0}\right).\vec{u_t}\right.\\ &\left.+\frac{a^2}{a_0}\frac{\partial A_x}{\partial a}\left(\frac{t-t_0}{a_0}, \frac{a}{a_0}\right).\vec{u_a}\right)\\ &= \left(\frac{a}{a_0}\right)^2\frac{\partial^2 A_x}{\partial t^2}\left(\frac{t-t_0}{a_0}, \frac{a}{a_0}\right)\\ &+\left(\frac{a}{a_0}\right)^2\frac{\partial^2 A_x}{\partial a^2}\left(\frac{t-t_0}{a_0}, \frac{a}{a_0}\right)\\ &+2\left(\frac{a}{a_0}\right)\frac{\partial A_x}{\partial a}\left(\frac{t-t_0}{a_0}, \frac{a}{a_0}\right)\\ &= \left.\frac{\partial A_x(t', a'); \tau)}{\partial \tau}\right|_{\substack{t' = (t-t_0).a_0^{-1}\\ a' = \frac{a}{a_0}.}}\end{aligned}$$

Therefore, using recurrence, the relation

$$A_{\mathcal{A}x}(t,a;\tau) = A_x\left(\frac{t-t_0}{a_0}, \frac{a}{a_0}; \tau\right),$$

holds and the processed representation is covariant with respect to the affine operator. Note that (6) is not the only diffusion scheme yielding bilinear representation covariant with respect to affine changes. Due to the affine affine invariance of the differential operator $a\partial/\partial a$, it can be factorized in other ways, leading to different diffusion equations, among which

$$\begin{aligned}\frac{\partial A_x(t,a;\tau)}{\partial \tau} &= a^2\,\mathrm{div}_{t,a}(\nabla_{t,a}A_x(t,a;\tau)),\\ \frac{\partial A_x(t,a;\tau)}{\partial \tau} &= a\,\mathrm{div}_{t,a}(a\nabla_{t,a}A_x(t,a;\tau)).\end{aligned}$$

We chose this scheme because of the preservation of the energy it ensures. Renouncing to the bilinearity and thus to the affine class, one can also use the affine diffusion described in [8].

Figure (1) illustrates our approach on a signal made of two gaussian atoms. One can see that the shape of the smoothing kernel for the affine diffusion depends on the frequency, and therefore on the scale, whereas it is constant for Wheyl diffusion. One can also notice the similarity between the scalogram and the affine diffusion for this signal. Note that these are time-frequency representations.

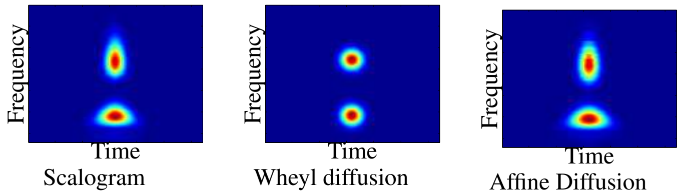

Fig. 1. Comparaison of different smoothings of two atoms.

3.3. Adaptive affine diffusion

Let us now use a conductance function $c_x(t, a)$ depending on both time-scale location and the analyzed signal to locally control the amount of smoothing. The general form for an adaptive affine diffusion is:

$$\begin{cases} A_x(t,a;\tau=0) = \Omega_x(t,a) \\ \frac{\partial A_x(t,a;\tau)}{\partial \tau} = \mathrm{div}_{t,a}(a^2 c_x(t,a)\nabla_{t,a}A_x(t,a;\tau)), \end{cases} \tag{7}$$

where A_x is the smoothed representation and Ω_x the representation to be processed. The divergence term can be expanded as follows

$$\begin{aligned}&\frac{\partial A_x(t,a;\tau)}{\partial \tau}\\ &= \mathrm{div}_{t,a}(c(t,a)a^2\nabla_{t,a}A_x(t,a;\tau))\\ &= \mathrm{div}_{t,a}\left(a^2 c_x(t,a)\left[\frac{\partial A_x}{\partial t}(t,a).\vec{u_t} + \frac{\partial A_x}{\partial a}(t,a).\vec{u_a}\right]\right)\\ &= a^2 c_x(t,a)\frac{\partial^2 A_x}{\partial t^2}(t,a) + a^2\frac{\partial c_x}{\partial t}(t,a)\frac{\partial A_x}{\partial t}(t,a)\\ &+a^2 c_x(t,a)\frac{\partial^2 A_x}{\partial a^2}(t,a) + 2a\,c_x(t,a)\frac{\partial A_x}{\partial a}(t,a)\\ &+a^2\frac{\partial c_x}{\partial a}(t,a)\frac{\partial A_x}{\partial a}(t,a).\end{aligned}$$

Conductance function $c_x(t, a)$ such that

$$\frac{\partial}{\partial t}c_{\mathcal{A}x}(t,a) = a_0^{-1}\frac{\partial c_x}{\partial t}\left(\frac{t-t_0}{a_0}, \frac{a}{a_0}\right), \tag{8}$$

$$\frac{\partial}{\partial a} c_{\mathcal{A}x}(t,a) = a_0^{-1} \frac{\partial c_x}{\partial a} \left(\frac{t - t_0}{a_0}, \frac{a}{a_0} \right), \qquad (9)$$

then ensures that the processed distribution satisfies the covariance property. One can check that conductance functions of the form $c_x(t,a) = c(I_x(t,a))$ where I_x denotes any representation of the affine class fulfill these requirements.

For the application in sight, namely increasing the readablity of time-scale representations, we can use the a priori information according to which spectrograms and scalograms usually do not suffer from cumbersome interferences. This idea was proposed in [4] for Wheyl diffusion: areas taking on high values in such distributions are associated with signal terms and have to be preserved during the diffusion, while areas taking on low values are likely to be interference terms and are to be smoothed. This technique will be referred to as adaptive diffusion because the action is different depending on the time-scale area to be processed. One can also use the a priori information according to which noise and interference generally tend to have no structure or thin structure compared to signal terms. Therefore, one can identify structures as areas taking on high values. Protecting these areas favors structured patterns while less structured ones are smoothed by the diffusion. It will be referred to as self controlled diffusion.

For both schemes, we can design a conductance function as follows

$$c_x(t,a;\tau) = \left(1 + \left(\frac{B_x(t,a)}{\delta} \right)^{\alpha} \right)^{-1} . \qquad (10)$$

Depending on the chosen scheme, $B_x(t,a)$ is either a scalogram or the processed diffusion itself at iteration τ. Parameters δ and α can be used to tune the behavior of the conductance function. Figure (2) illustrates the ability of adaptive affine diffusion to improve readability of a representation.

As for the Wheyl diffusion, one has to find a stopping criterion to stop this iterative process. Because the criterion proposed in [4], relying of the entropy of the diffused representations, does not depend on covariance properties, we suggest referring to this paper for the question.

4. CONCLUSION

We have presented an adaptive and iterative smoothing scheme for time-scale representations of the affine class. We have shown that it preserves the covariance property in homogeneous and non-homogeneous cases, providing a locally adapted smoothing for the affine class. In the homogeneous case, the bilinearity of the processed distribution ensures it can be written as an affine convolution (3). Because of the versatility of the conductance function technique, this diffusion scheme can be used within many contexts. We have presented an example aiming at improving the readability of representations. We have provided two conductance functions and illustrated their efficiency.

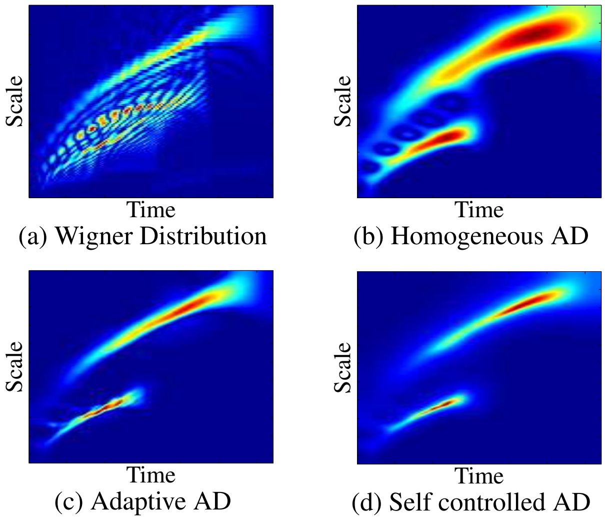

Fig. 2. The two conductance functions succeed in providing a sharp, concentrated and clean representation for this signal composed of two power laws.

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