ON EQUIVALENCE BETWEEN DETECTORS OBTAINED FROM SECOND-ORDER MEASURES OF PERFORMANCE

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ABSTRACT

Second-order measures of performance are often used for deriving detectors. Many of these are related or equivalent. In this paper we propose unifying views of these criteria that depend on the approach used to design detectors. In the case of deriving unconstrained detectors, we show that second-order criteria can be classified into two categories depending on their ability to yield a one-to-one function of the likelihood ratio. Criteria that satisfy this fundamental property are logically called relevant criteria. If constraints are imposed on the detection structure, two relevant criteria may lead to non-equivalent detectors in the sense that the receiver operating characteristic is not the same. We then have to reconsider our strategy for partitioning the set of second-order measures of performance. For practical reasons, this problem is addressed in the case of linear detection structures. Within this context, we propose a necessary and sufficient condition under which distinct criteria provide equivalent detectors. Finally, results are illustrated by considering well-known second-order criteria.

1. INTRODUCTION

The purpose of detection is to determine to which of two classes ω_0 or ω_1 a given observation X belongs. Statistical detection theories lead to the fundamental result that the optimum test consists in comparing any strictly monotonic function of the likelihood ratio to a threshold value [1]. Implementing such a detector may be intractable or impossible, because of incomplete specification of the underlying probabilistic model. Amongst the myriad of alternative design criteria that have been proposed [2], second-order measures of performance are widely used [3]. Let

$$S(X) \underset{\omega_0}{\overset{\omega_1}{\gtrless}} 0. \qquad (1)$$

be a given decision rule. These criteria are defined in terms of first and second-order moments of the decision statistic $S(X)$, namely

$$m_i \triangleq \mathrm{E}\{S \,|\, \omega_i\}, \qquad \sigma_i^2 \triangleq \mathrm{Var}\{S \,|\, \omega_i\}, \qquad (2)$$

with $i \in \{0,1\}$. Many contributions can be found in the literature to justify the use of particular criteria of this family, such as generalized signal-to-noise ratio, deflection and Fisher criterion (see e.g. [3], [4] and [5]).

This paper deals with equivalence between detection structures designed from second-order criteria. Our purpose is to show that their exist classes of criteria that lead to equivalent detectors in the sense that the receiver operating characteristic (ROC) is the same. First, we consider the case of unconstrained detectors design and we investigate the family of second-order criteria that guaranty the best solution is the sense of classical decision theories. Next we address the problem of constrained detectors design. In the particular case of linear decision rule, we give a necessary and sufficient condition under which distinct criteria provide equivalent detectors. Both approaches are illustrated with well-known criteria such as the generalized signal-to-noise ratio and mean square error. Finally we present some concluding remarks.

2. DESIGN OF UNCONSTRAINED DETECTORS USING SECOND-ORDER CRITERIA

Let Ψ be the family of second-order performance measure and $\psi(m_0, m_1, \sigma_0^2, \sigma_1^2) \in \Psi$. Let $S(X)$ be the optimal discriminant function $S(X)$ that results from the optimization of $\psi(m_0, m_1, \sigma_0^2, \sigma_1^2)$. The ability of this criterion to provide an equivalent statistic to the likelihood ratio is not always guaranteed. In this section, we show that the whole set of second-order criteria Ψ can be partitioned into two subsets, as illustrated in Fig.1. The first subset, denoted Ψ_{R}, corresponds to criteria which are relevant in the sense that their optimization with respect to $S(X)$ yields a one-to-one function of the likelihood ratio. The second subset Ψ_{NR} contains criteria that are non-relevant. These two categories of criteria are characterized below.

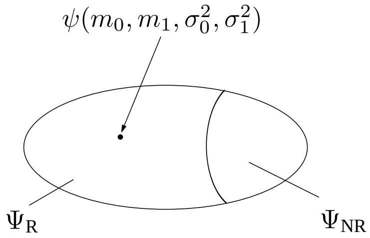

Fig. 1. Partitioning of the class Ψ of second-order measures. The subset Ψ_{R} corresponds to criteria that yield one-to-one functions of the likelihood ratio.

2.1. Characterization of relevant second-order criteria

In [6, 7], a necessary and sufficient condition on second-order criteria has been proposed to guarantee a decision statistic equivalent to the likelihood ratio. The authors have shown that $\psi(m_0, m_1, \sigma_0^2, \sigma_1^2) \in \Psi_{\mathrm{R}}$ if and only if it satisfies:

$$(m_1 - m_0)\,\frac{\partial \psi}{\partial \sigma_0^2}\,\frac{\partial \psi}{\partial \sigma_1^2} + \frac{1}{2}\left(\frac{\partial \psi}{\partial \sigma_1^2}\,\frac{\partial \psi}{\partial m_0} - \frac{\partial \psi}{\partial \sigma_0^2}\,\frac{\partial \psi}{\partial m_1}\right) \neq 0. \quad (3)$$

The above equation was obtained by considering that the decision statistic $S(X)$ resulting from the optimisation of $\psi(m_0, m_1, \sigma_0^2, \sigma_1^2)$ must be a strictly monotonic function of the likelihood ratio $L(X)$. This condition characterizes criteria to be used in practice. In the next section, we determine to which of the two classes Ψ_{NR} or Ψ_{R} some well known criteria belong.

2.2. Case of the mean-square error

Physical interpretation of mean-square error (MSE) justifies its popularity. Let $\gamma(X)$ be the desired output of the detector to be designed. The MSE between current and desired outputs is [3]:

$$\begin{aligned}\psi_{\mathrm{MSE}} &= E\{(S(X) - \gamma(X))^2\} \\ &= E\{S^2(X)\} - 2E\{\gamma(X)S(X)\} + E\{\gamma^2(X)\}.\end{aligned}$$

There exist many possible functional forms of $\gamma(X)$. As an example, one can choose $\gamma(X) = -1$ if $X \in \omega_0$, and $\gamma(X) = +1$ if $X \in \omega_1$. This is a reasonable choice since in (1) the discriminant function $S(X)$ is supposed to be negative if $X \in \omega_0$ and positive if $X \in \omega_1$.

We will now express the MSE as a function of the conditional moments m_i and σ_i^2 of $S(X)$. The first term $E\{S^2(X)\}$ in the definition of ψ_{MSE} mentioned above is the second-order moment of $S(X)$. It is given by

$$E\{S^2(X)\} = P_0(\sigma_0^2 + m_0^2) + P_1(\sigma_1^2 + m_1^2), \quad (4)$$

where P_i is the *a priori* probability of ω_i. The second term can be expressed as

$$\begin{aligned}E\{\gamma(X)S(X)\} &= P_0 E\{(-1)S(X)|\omega_0\} \\ &\quad + P_1 E\{(+1)S(X)|\omega_1\} \quad (5) \\ &= -P_0 m_0 + P_1 m_1.\end{aligned}$$

Substituting (4) and (5) into the definition of ψ_{MSE} gives:

$$\begin{aligned}\psi_{\mathrm{MSE}} &= P_0(\sigma_0^2 + m_0^2) + P_1(\sigma_1^2 + m_1^2) \\ &\quad - 2(-P_0 m_0 + P_1 m_1) + E\{\gamma^2(X)\}. \quad (6)\end{aligned}$$

The term $E\{\gamma^2(X)\}$ does not depend on $S(X)$, which implies that it is independent of m_i and σ_i^2. Thus, (6) shows that $\psi_{\mathrm{MSE}} \in \Psi$ since it depends only on m_i and σ_i^2. Let us now study the relevance of this criterion using (3). Calculating the first order derivatives of ψ_{MSE} with respect to m_i and σ_i^2 yields:

$$\frac{\partial \psi_{\mathrm{MSE}}}{\partial m_0} = 2P_0 m_0 + 2P_0 \qquad \frac{\partial \psi_{\mathrm{MSE}}}{\partial m_1} = 2P_1 m_1 - 2P_1 \quad (7)$$

$$\frac{\partial \psi_{\mathrm{MSE}}}{\partial \sigma_0^2} = P_0 \qquad \frac{\partial \psi_{\mathrm{MSE}}}{\partial \sigma_1^2} = P_1. \quad (8)$$

Inserting these results into (3) provides the condition:

$$2P_0 P_1 \neq 0 \quad (9)$$

This expression is satisfied for all X. Thus the MSE is a relevant second-order criterion.

2.3. Case of the generalized signal-to-noise ratio

In this subsection, we are concerned with another second-order criterion that is frequently used in practice: the generalized signal-to-noise ratio (GSNR). It is defined as [3]

$$\psi_{\mathrm{GSNR}_\alpha} = \frac{(m_1 - m_0)^2}{\alpha\sigma_1^2 + (1-\alpha)\sigma_0^2}, \quad (10)$$

where m_i and σ_i^2 are defined as in (2), and α is a parameter in $[0, 1]$. The criterion $\psi_{\mathrm{GSNR}_\alpha}$ is a measure of clustering for the two competing classes ω_0 and ω_1: it tends to be large when the between-class scatter $(m_1 - m_0)^2$ increases and the within-class scatter $\alpha\sigma_1^2 + (1-\alpha)\sigma_0^2$ decreases. Properties of the GSNR have been extensively studied [3, 2, 4]. In particular, its relevancy has been proved via ad'hoc approaches. In order to illustrate the generic condition (3), we now propose to verify one more time that the GSNR is a relevant criterion. Calculating the first order derivatives of $\psi_{\mathrm{GSNR}_\alpha}$ and inserting them in (3), we obtain that the GSNR is a relevant criterion if and only if $\alpha(1-\alpha)(m_1 - m_0)^2 + \alpha\sigma_1^2 + (1-\alpha)\sigma_0^2$ is not identically equal to 0 for all X. Obviously, this condition is satisfied.

3. DESIGN OF CONSTRAINED DETECTORS USING SECOND-ORDER CRITERIA

In [7], the authors have shown that the decision statistic $S(X)$ that maximizes any given second-order criterion $\psi(m_0, m_1, \sigma_0^2, \sigma_1^2)$ is of the form:

$$S(X) = -\frac{1}{2}\frac{\frac{\partial\psi}{\partial m_0} + \frac{\partial\psi}{\partial m_1}L(X)}{\frac{\partial\psi}{\partial\sigma_0^2} + \frac{\partial\psi}{\partial\sigma_1^2}L(X)} + \frac{m_0\frac{\partial\psi}{\partial\sigma_0^2} + m_1\frac{\partial\psi}{\partial\sigma_1^2}L(X)}{\frac{\partial\psi}{\partial\sigma_0^2} + \frac{\partial\psi}{\partial\sigma_1^2}L(X)}. \qquad (11)$$

Note that the condition (3) has been obtained by imposing the monotony condition on $S(X)$ given above with respect to the likelihood ratio $L(X)$. In many applications, obtaining $S(X)$ or $L(X)$ may be an intractable problem because of incomplete specifications of conditional probability densities. Consequently, the strategy that follows is often used for deriving receivers [2]:

1. selecting a class $\mathcal{C}$ of constrained detectors;
2. tempting to pick the detector of $\mathcal{C}$ that optimizes a given measure of performance.

Due to the simplicity and robustness of linear detectors, many attempts have been made to determine the best one for a given problem via the optimization of a second-order criterion. These decision rules are of the form:

$$S(X) = K^T X + s \underset{\omega_0}{\overset{\omega_1}{\gtrless}} 0. \qquad (12)$$

Here, $K = [k_1 \ldots k_n]^T$ represents the direction onto which any n-dimensional observation X is projected, and s is the detector threshold. The conditional expected values and variances of $S(X)$ are given by

$$m_i = \mathrm{E}\{S \,|\, \omega_i\} = K^T M_i + s \qquad (13)$$
$$\sigma_i^2 = \mathrm{Var}\{S \,|\, \omega_i\} = K^T \Sigma_i K, \qquad (14)$$

where M_i and Σ_i are the conditional expected vectors and covariance matrices of the observation X. Optimization of $S(X)$ is achieved by equating to zero the partial derivatives of ψ with respect to K and s. Using the fact that any constant term multiplying to K can be eliminated leads directly to the following proposition (see [5, pp. 133-4] and [8]).

Proposition 1. *Let $S(X) \triangleq K^T X + s$ be any linear decision statistic. The optimum projection vector K under which the maximum value of any second-order criteria ψ is reached satisfies*

$$[\rho\Sigma_0 + (1-\rho)\Sigma_1]\, K = M_1 - M_0, \qquad (15)$$

where M_i and Σ_i are the conditional expected vectors and covariance matrices of X. The parameter ρ depends on the criterion ψ as follows:

$$\rho = \frac{\frac{\partial\psi}{\partial\sigma_0^2}}{\frac{\partial\psi}{\partial\sigma_0^2} + \frac{\partial\psi}{\partial\sigma_1^2}}. \qquad (16)$$

In the next section, we use Proposition 1 to give a condition under which two distinct second-order criteria ψ_1 and ψ_2 provide equivalent linear detectors in the sense that their receiver operating characteristics (ROC) are equal.

3.1. Second-order criteria leading to equivalent linear detectors

In Section 2.1, we have shown that, in the unconstrained case, all the criteria that satisfy (3) lead to equivalent detectors since the resulting detection statistic are strictly monotonic functions of the likelihood ratio. However, if constraints are imposed on the structure of $S(X)$, two relevant criteria may lead to non-equivalent detectors. The next proposition considers the case of linear detectors.

Proposition 2. *Let ψ_1 and ψ_2 be any second-order criteria. Optimizing ψ_1 and ψ_2 leads to equivalent linear detectors in the sense that their ROC are equal if, and only if,*

$$\rho_1 \triangleq \frac{\frac{\partial\psi_1}{\partial\sigma_0^2}}{\frac{\partial\psi_1}{\partial\sigma_0^2} + \frac{\partial\psi_1}{\partial\sigma_1^2}} = \frac{\frac{\partial\psi_2}{\partial\sigma_0^2}}{\frac{\partial\psi_2}{\partial\sigma_0^2} + \frac{\partial\psi_2}{\partial\sigma_1^2}} \triangleq \rho_2. \qquad (17)$$

This condition directly results from the fact that s does not have any effect on the ROC of the detection structure (15), and ρ in (16) is the only parameter that influences the direction of the projection vector K.

Note that (15) is also valid for any non-relevant criterion. This implies that two linear detectors, parameterized by K_1 and K_2 resulting from the optimization of the two different criteria ψ_1 and ψ_2 with $\psi_1 \in \Psi_{\mathrm{R}}$ and $\psi_2 \in \Psi_{\mathrm{NR}}$, can be equivalent. As an example, let us consider the following second-order criterion, which has no tangible physical meaning: $\psi = (m_1 - m_0)^2 + \alpha\sigma_1^2 + (1-\alpha)\sigma_0^2$. Since it does not satisfy Condition (3), ψ is non-relevant. However, it leads to linear detectors that are equivalent to those provided by the GSNR since ψ and $\psi_{\mathrm{GSNR}_\alpha}$ both satisfy Proposition 2.

3.2. Particular cases of the MSE and GSNR

We shall now determine the classes of second-order criteria which lead to linear detectors that are equivalent to those provided by ψ_{MSE} and $\psi_{\mathrm{GSNR}_\alpha}$. Proposition 1 shows that the optimum vector K_{MSE} under which a minimum of the MSE is reached satisfies:

$$[\rho_{\mathrm{MSE}}\Sigma_0 + (1-\rho_{\mathrm{MSE}})\Sigma_1]\, K_{\mathrm{MSE}} = M_1 - M_0, \qquad (18)$$

where the parameter ρ_{MSE} given below is obtained by combining (8) and (16):

$$\rho_{\text{MSE}} = P_0. \tag{19}$$

Using Condition (17) with ψ_{MSE} and any other second-order criterion ψ yields

$$\frac{\frac{\partial\psi}{\partial\sigma_0^2}}{\frac{\partial\psi}{\partial\sigma_0^2} + \frac{\partial\psi}{\partial\sigma_1^2}} = P_0, \tag{20}$$

which is equivalent to:

$$P_1 \frac{\partial\psi}{\partial\sigma_0^2} = P_0 \frac{\partial\psi}{\partial\sigma_1^2}. \tag{21}$$

The equation mentioned above can be solved simply by posing $\psi \triangleq \psi(m_0, m_1, u, v)$ with $u = P_0\sigma_0^2 + P_1\sigma_1^2$ and $v = P_0\sigma_0^2 - P_1\sigma_1^2$. We then obtain that solutions of (21) are of the form $\psi(m_0, m_1, P_0\sigma_0^2 + P_1\sigma_1^2)$. This result leads to the following proposition.

Proposition 3. *Let ψ be any second-order criteria. Optimizing ψ and ψ_{MSE} leads to equivalent linear detectors in the sense that their ROC are equal if, and only if, ψ is of the form $\psi(m_0, m_1, P_0\sigma_0^2 + P_1\sigma_1^2)$.*

The above condition means that the criterion ψ must depend only on σ_0^2 and σ_1^2 through $P_0\sigma_0^2 + P_1\sigma_1^2$. This directly implies that the GSNR and the MSE provide equivalent linear detectors if $\alpha = P_1$.

Consider now the case of the GSNR. Given $\alpha \in [0, 1]$, one can calculate from (15) and (16) the projection vector K_{GSNR} under which the maximum value of the GSNR is achieved:

$$[(1-\alpha)\Sigma_0 + \alpha\Sigma_1]\, K_{\text{GSNR}} = M_1 - M_0. \tag{22}$$

With the same calculation as above, one can determine criteria ψ which lead to linear detectors that are equivalent to the detector provided by the maximization of $\psi_{\text{GSNR}_\alpha}$. The following result is finally obtained:

Proposition 4. *Let ψ be any second-order criteria. Optimizing ψ or ψ_{GSNR_α} leads to equivalent linear detectors in the sense that their ROC is the same if, and only if, ψ is of the form $\psi(m_0, m_1, \alpha\sigma_1^2 + (1-\alpha)\sigma_0^2))$.*

4. CONCLUSION

Second-order criteria are largely used in statistical detection. Many of these are related or equivalent. Therefore, we have presented in this paper unifying views of these measures of performance, depending on the approach used for designing detectors.

In the case of designing unconstrained detectors, we have shown that the family of second-order criteria can be partitioned into two subsets. These subsets correspond to criteria which are relevant or not in the sense that their optimization yields a one-to-one function of the likelihood ratio or not. This has been illustrated by an analysis of the relevance of the generalized signal-to-noise ratio and mean square error.

If constraints are imposed on the detection structure, two relevant criteria may lead to non-equivalent detectors in the sense that their ROC are different. We then have to reconsider the partitioning strategy. For practical reasons, the discussion has been limited to linear detectors. Within this context, we have proposed a necessary and sufficient condition under which distinct criteria provide equivalent detectors. This has been illustrated by determining criteria which lead to linear detectors that are equivalent to the detectors maximizing the generalized signal-to-noise ratio or mean square error. An extension of this approach to other detection structures will be considered in future work.

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